

The University of Chicago

FOUNDED BY JOHN D. ROCKEFELLER

On the Determination of the Ternary Modular Groups

A DISSERTATION

SUBMITTED TO THE FACULTY

OF THE

OGDEN GRADUATE SCHOOL OF SCIENCE

IN CANDIDACY FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

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ROBERT LACEY BÖRGER

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On the Determination of the Ternary Modular Groups.

By R. L. BÖRGER.

Introduction.

The subgroups of the linear fractional group $z' = \frac{\alpha z + \beta}{\gamma z + \delta}$ in $GF[p^n]$ have been determined.* Closely related to this is the determination of the binary groups of determinant unity in the $GF[p^n]$.† The subgroups of the special ternary group $GF[p]$ have been found.‡ The object of the present paper is to apply the method of the paper last cited to the group in $GF[p^2]$. The order of the special ternary linear group in $GF[p^2]$ is $p^6(p^6 - 1)(p^4 - 1)$. The subgroups of order a multiple of p , $p^i M$ ($i = 6, 5, 4$) (M prime to p) are determined in this paper. The determination of the groups of these high orders applies in particular to the question of the minimum index of a subgroup of total group. For $i = 3, 2, 1$ the method can not be employed and recourse must be had to special devices.

Subgroups of Order a Power of p .

1. The order of the special linear ternary group G in the $GF[p^2]$ is $p^6(p^6 - 1)(p^4 - 1)$. Within G every subgroup of order a power of p is conjugate with a subgroup of the group G_{p^6} of the operations

$$[\alpha] = \begin{pmatrix} 1 & 0 & 0 \\ \alpha_{21} & 1 & 0 \\ \alpha_{31} & \alpha_{32} & 1 \end{pmatrix} \quad (1)$$

with the α_{ij} arbitrary in the $GF[p^2]$.

Since§ the commutator of G_{p^6} is of order p^2 and every operation is of

* Gierster, Wiman; Moore, *University of Chicago Decennial Publications*, Vol. IX (1903), pp. 141-190; Dickson: "Linear Groups."

† Made by Dickson — unpublished.

‡ Dickson, *AMERICAN JOURNAL*, Vol. XXVII, pp. 189-202.

§ The results in §§ 1-3 are proved in detail in *AMERICAN JOURNAL*, Vol. XXVII, pp. 289-293.

order p , ($p \neq 2$), there are $\frac{p^4-1}{p-1}$ subgroups of order p^5 . They are given by a linear relation modulo p ,

$$l(a_{210}, a_{211}, a_{320}, a_{321}) \equiv 0. \quad (2)$$

2. The commutator subgroup K of any group L defined by a relation (2) is of order p^2 .

THEOREM: Any group L_{p^5} defined by one linear relation (2) has exactly $p^2 + p + 1$ subgroups of order p^4 , defined by two independent relations (2). The number of distinct subgroups of order p^4 of G_{p^5} is thus

$$p^4 + p^3 + 2p^2 + p + 1.$$

They are as follows:

$$(I_a) \quad a_{210} = g a_{320} + h a_{321}, \quad a_{211} = l a_{320} + m a_{321} \quad [\alpha_{21} = t \alpha_{32}].$$

$$(I) \quad a_{210} = g a_{320} + h a_{321}, \quad a_{211} = l a_{320} + m a_{321} \quad [\alpha_{21} \neq t \alpha_{32}].$$

$$(II) \quad a_{321} = h a_{320}, \quad a_{211} = l a_{210} + m a_{320}.$$

$$(III) \quad a_{321} = h a_{320}, \quad a_{210} = m a_{320}.$$

$$(IV) \quad a_{320} \equiv 0, \quad a_{211} = l a_{211} + m a_{321}.$$

$$(V) \quad a_{320} \equiv 0, \quad a_{210} = m a_{321}.$$

$$(VI) \quad a_{320} \equiv 0, \quad a_{321} \equiv 0.$$

THEOREM. The only commutative subgroups of order p^4 are those defined by $\{\alpha_{32} = 0\}$ and $\{\alpha_{21} = t \alpha_{32}\}$. For the remaining subgroups of order p^4 the commutator subgroup is of order p .

3. THEOREM. For $p > 2$ the subgroups G_{p^3} of the G_{p^4} , given in the first column, are obtained by annexing to the two relations defining G_{p^4} an independent linear homogeneous relation between the elements given in the second column.

G_{p^4} .

ELEMENTS.

$$(I_a) \quad a_{320}, \quad a_{321}, \quad a_{310} - \frac{1}{2} g a_{320} - h a_{320} a_{321} - \frac{1}{2} (v g + \mu h) a_{320}^2, \\ a_{311} - \frac{h}{2v} a_{321}^2 - \left(g + \mu \frac{h}{v}\right) a_{320} a_{321} - \frac{1}{2} \left(h + \frac{\mu^2 h}{v} + \mu g\right) a_{321}^2.$$

$$(I) \quad a_{320}, \quad a_{321}, \quad (m - g - l\mu) a_{310} - (h - v l) a_{311} - u a_{320}^2 - v a_{320} a_{321} - w a_{321}^2.$$

$$(II) \quad a_{320}, \quad a_{210}, \quad (l + h + l h \mu) a_{310} - (1 + h l v) a_{311} + \frac{1}{2} m (1 + h \mu - h^2 v) a_{320}^2.$$

$$(III) \quad a_{320}, \quad a_{211}, \quad (1 + h \mu) a_{310} - h v a_{311} - \frac{m}{2} (1 + h \mu - h^2 v) a_{320}^2.$$

$$(IV) \quad a_{321}, \quad a_{210}, \quad (1 + l \mu) a_{310} - l v a_{311} - \frac{1}{2} m v a_{321}^2.$$

$$(V) \quad a_{321}, \quad a_{211}, \quad \mu a_{310} - v a_{311} + \frac{1}{2} m v a_{321}^2.$$

$$(VI) \quad a_{210}, \quad a_{211}, \quad a_{310}, \quad a_{311}.$$

The u , v and w used in (I) denote

$$\begin{aligned} u &\equiv \frac{1}{2} g(m - g - l\mu) - \frac{1}{2} l(h - v l), \\ v &\equiv m v l - h g - h l \mu, \\ w &\equiv \frac{1}{2} m v(m - g - l\mu) - \frac{1}{2} (h - l v)(h + m\mu). \end{aligned}$$

The Galois Field is defined by the irreducible congruence $\rho^2 \equiv \mu \rho + v \pmod{p}$.

4. The subgroups of order p^2 of any abelian group G_{p^3} represented in the first column are given by annexing, to the relations defining the G_{p^3} , an independent linear homogeneous relation between the elements of the second column.

G_{p^3}	ELEMENTS.
(I _a)	$a_{320}, a_{321}, a_{310} - \frac{1}{2} g a_{320}^2 - h a_{320} a_{321} - \frac{1}{2} (v g + \mu h) a_{321}^2,$ $a_{311} - \frac{h}{2 v} a_{321}^2 - \left(g + \frac{\mu h}{v}\right) a_{320} a_{321} - \frac{1}{2} \left(h + \frac{\mu^2 h}{v} + \mu g\right) a_{321}^2.$
I _{1,3}	$a_{320}, a_{310} - \frac{1}{2} g a_{320}^2, a_{311} - \frac{1}{2} l a_{320}^2.$
I _{2,3}	$a_{321}, a_{310} - \frac{v h}{2} a_{321}^2, a_{311} - \frac{1}{2} (h + \mu m) a_{321}^2.$
I ₃	$a_{320}, a_{310} - \frac{1}{2} \lambda a_{320}^2, a_{311} - \frac{1}{2} \eta a_{320}^2.$
II _{1,3}	$a_{320}, a_{310} - \frac{1}{2} v h m a_{320}^2, a_{311} - \frac{1}{2} (m + \mu h m) a_{320}^2.$
II _{2,3}	$a_{210}, a_{310}, a_{311}.$
II ₃	$a_{320}, a_{310} - \frac{1}{2} [k + v h(lk + m)] a_{320}^2, a_{311} - \frac{1}{2} \{lk + m + h k + \mu h(lk + m)\} a_{320}^2.$
III _{1,3}	$a_{320}, a_{310} - \frac{1}{2} m a_{320}^2, a_{311} - \frac{h m}{2} a_{320}^2.$
III _{2,3}	$a_{211}, a_{310}, a_{311}.$
III ₃	$a_{320}, a_{311} - \frac{1}{2} (m + v h k) a_{320}^2, a_{311} - \frac{1}{2} (k + h m + \mu h k) a_{320}^2.$
IV _{1,3}	$a_{321}, a_{310} - \frac{1}{2} m v a_{321}^2, a_{311} - \frac{1}{2} m v a_{321}^2.$
IV _{2,3}	$a_{210}, a_{310}, a_{311}.$
IV ₃	$a_{321}, a_{310} - \frac{v}{2} (k l + m) a_{321}^2, a_{311} - \frac{m v}{2} a_{321}^2.$
V _{1,3}	$a_{321}, a_{310}, a_{311} - \frac{1}{2} m a_{321}^2.$
V _{2,3}	$a_{211}, a_{310}, a_{311}.$
V ₃	$a_{321}, a_{310} - v \frac{k}{2} a_{321}^2, a_{311} - \frac{1}{2} (m + m k) a_{321}^2.$
VI	$a_{210}, a_{211}, a_{310}, a_{311}.$ $[\lambda \equiv g + h k + v k l + v k^2 m, \quad \eta \equiv l + m k + g k + h k^2 + \mu k l + \mu k^2 m,$ $k \equiv -l_1/l_2.]$

The notation in first column meaning that coefficients corresponding to the subscripts are 0, the others being distinct from 0. Thus: V_{1,3} means that

linear relation defining the G_{p^3} has $l_1 = l_3 = 0$, $l_2 \neq 0$; V_3 means that linear relation defining the G_{p^3} has $l_3 = 0$, $l_1 \neq 0$, $l_2 \neq 0$.

The remaining subgroups of order p^3 are non-abelian.

*Lemma.** The commutator subgroup of any non-abelian group of order p^3 is of order p .

Proof: If G_p is a subgroup of invariant operations, the quotient group G_{p^3}/G_p is of order p^2 . Hence G_p contains the commutator subgroup of G_{p^3} . Since G_{p^3} is non-abelian the commutator is G_p .

It follows that the number of subgroups of order p^2 of G_{p^3} (G_{p^3} being non-abelian and operations all of order p) is $p + 1$. They are given by annexing to the relations defining G_{p^3} an independent linear homogeneous relation between the elements of the first two columns in § 4.

The subgroups of order p are easily found. They will not be written here.

Subgroups of Order a Multiple of p .

5. *Lemma I.* Every operation (β_{ij}) permutable with a subgroup, not the identity, of G_{p^6} is contained in the set:

$$\begin{pmatrix} \beta_{11} & 0 & 0 \\ \beta_{21} & \beta_{22} & 0 \\ \beta_{31} & \beta_{32} & \beta_{33} \end{pmatrix}; \begin{pmatrix} \beta_{11} & \beta_{12} & 0 \\ \beta_{21} & \beta_{22} & 0 \\ \beta_{31} & \beta_{32} & \beta_{33} \end{pmatrix}; \begin{pmatrix} \beta_{11} & 0 & 0 \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \end{pmatrix}. \quad (3)$$

Proof: Among the conditions for $(\beta_{ij})[\alpha] = [\gamma](\beta_{ij})$ are:

$$\beta_{11} + \beta_{12}\alpha_{21} + \beta_{13}\alpha_{31} = \beta_{11}, \quad (1)$$

$$\beta_{21} + \beta_{22}\alpha_{21} + \beta_{23}\alpha_{31} = \beta_{11}\gamma_{21} + \beta_{21}, \quad (2)$$

$$\beta_{12} + \beta_{13}\alpha_{32} = \beta_{12}, \quad (3)$$

$$\beta_{22} + \beta_{23}\alpha_{32} = \beta_{12}\gamma_{21} + \beta_{22}, \quad (4)$$

$$\beta_{32} + \beta_{33}\alpha_{32} = \beta_{12}\gamma_{31} + \beta_{22}\gamma_{32} + \beta_{32}, \quad (5)$$

$$\beta_{23} = \beta_{23} + \gamma_{21}\beta_{13}. \quad (6)$$

If $\alpha_{21} \equiv 0$, $\alpha_{32} \neq 0$, $\alpha_{31} \neq 0$, then $\beta_{13} = \beta_{23} = 0$ by (1) and (2).

In like manner we treat the remaining cases and prove the lemma.

Lemma II. The only factors $\equiv 1 \pmod{p^4}$ of $(p^6 - 1)(p^4 - 1) \equiv \omega$ are 1 and ω . The proof is similar to that in the paper† referred to and will not be given here.

* Miller, *Bulletin American Mathematical Society*, 1897-98, p. 137.

† Dickson, *AMERICAN JOURNAL*, Vol. XXVII, p. 191.

Lemma III. Any binary transformation $B = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$, $\gamma \neq 0$, and all the $E_\lambda = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}$ generate every binary transformation T of determinant unity.

For proof, see reference above.

Lemma IV. If a group G_1 , of order $p^\beta N$ (N prime to p), contains a subgroup of G_{p^α} ($\alpha > \beta$) of order p^β and if P is an operation of G_1 permutable with G_{p^α} , then P is permutable with G_{p^β} .

Proof: Suppose P were not permutable with G_{p^β} . It would transform G_{p^β} into G'_{p^β} ; and $\{G_{p^\beta}, G'_{p^\beta}\}$, being of order a power of p , would be of order $p^{2\beta-c}$ (where p^c is the order of the greatest subgroup common to G_{p^β} and G'_{p^β}). But for all values of $c < \beta$ we have $p^{2\beta-c} > p^\beta$. Hence $c = \beta$ and $G'_{p^\beta} \equiv G_{p^\beta}$.

Lemma V. The operations permutable with every group G_{p^3} defined by

$$\alpha_{21} = 0, \quad k a_{321} + h a_{320} = 0$$

are contained in the set (3_1) .

Proof: By Lemma I, they are contained in (3_2) . Transforming G_{p^3} by (β_{ij}) ($\beta_{13} = \beta_{23} = 0$) it becomes $G'_{p^3}[\gamma_{ij}]$, having

$$\gamma_{31} \equiv \frac{\beta_{11}}{\beta_{33}} \alpha_{31} + \frac{\beta_{21}}{\beta_{33}} \alpha_{32}, \quad \gamma_{32} \equiv \frac{\beta_{12}}{\beta_{33}} \alpha_{31} + \frac{\beta_{22}}{\beta_{33}} \alpha_{32}.$$

Call

$$\begin{aligned} \frac{\beta_{11}}{\beta_{33}} &\equiv c_{11} + \rho d_{11}, & \frac{\beta_{22}}{\beta_{33}} &\equiv c_{22} + \rho d_{22}, \\ \frac{\beta_{12}}{\beta_{33}} &\equiv c_{12} + \rho d_{12}, & \frac{\beta_{21}}{\beta_{33}} &\equiv c_{21} + \rho d_{21}. \end{aligned}$$

Then $\gamma_{31} \equiv (c_{12} a_{310} + c_{22} a_{320} + \nu d_{12} a_{311} + \nu d_{22} a_{321})$
 $+ \rho (d_{12} a_{310} + c_{12} a_{311} + \mu d_{12} a_{311} + \mu d_{22} a_{321} + c_{22} a_{321} + d_{22} a_{320}).$

Setting $\alpha_{32} = 0$ and recalling the definition of G_{p^3} , we get the relation:

$$(c_{12} h + d_{12} k) a_{310} + (\nu d_{12} h + c_{12} k + \mu d_{12} k) a_{311} = 0.$$

Since a_{310} and a_{311} are arbitrary, we have

$$c_{12} h + d_{12} k = 0, \quad c_{12} k + d_{12} (\nu h + \mu k) = 0.$$

Thus $c_{12} = d_{12} = 0$, since the determinant of their coefficients equals $\nu h^2 + \mu h k - k^2$, which vanishes only for the excluded case $h = k = 0$, since $\rho^2 \equiv \mu \rho + \nu$ was assumed irreducible.

6. Let H be a subgroup of order $p^6 N$ (N prime to p) normalized to contain G_{p^6} . Then, by methods similar to those employed in the paper referred to, we derive the following

THEOREM. Within G every subgroup H of order a multiple of p^6 is conjugate with one of the following:

- a) The group of all the $p^6 f.g$ operations (3_1) with $\alpha_{11}^f = 1$, $\alpha_{22}^g = 1$,
 $\alpha_{33} = \alpha_{11}^{-1} \alpha_{22}^{-1}$.
- b) The group of all the $f p^6 (p^4 - 1)$ operations (3_2) with $\alpha_{33}^f = 1$,
 $\alpha_{11} \alpha_{22} - \alpha_{12} \alpha_{21} = \alpha_{33}^{-1}$.
- c) The group of all the $f p^6 (p^4 - 1)$ operations (3_3) with $\alpha_{11}^f = 1$,
 $\alpha_{22} \alpha_{33} - \alpha_{23} \alpha_{32} = \alpha_{11}^{-1}$, where f and g are any divisors of $p^2 - 1$.

7. Let H be a subgroup of G of order $p^5 N$ (N prime to p) normalized to contain a subgroup G_{p^5} of G_{p^6} . Let the G_{p^5} be defined by

$$l_1 a_{210} + l_2 a_{211} + l_3 a_{320} + l_4 a_{321} = 0.$$

(I). We exclude for the present the special case $l_1 = l_2 = 0$ and the special case $l_3 = l_4 = 0$ (treated under II and III below), and treat here the general case.

The number of conjugates to G_{p^5} is $1 + kp$, $1 + kp^2$, $1 + kp^3$, or $1 + kp^4$ (k being prime to p). We treat these cases in turn.

(I₁). Let the number of conjugates to G_{p^5} be $1 + kp$. Then two groups of order p^5 must contain a common subgroup of order p^{4*} and, by a theorem† due to Frobenius and Burnside independently, H must contain an operation P permutable with G_{p^4} but not with G_{p^5} . Since neither $l_1 = 0$, $l_2 = 0$ nor $l_3 = 0$, $l_4 = 0$ in the case before us, this common G_{p^4} must have neither $\alpha_{21} \equiv 0$ nor $\alpha_{32} \equiv 0$. Therefore the operations permutable with G_{p^4} are contained in (3_1) and P is also permutable with G_{p^5} . Hence P must, by Lemma IV, be permutable with G_{p^5} , in contradiction with the Frobenius-Burnside Theorem. Hence the number of conjugates to G_{p^5} can not be $1 + kp$.

(I₂). Suppose the number of conjugates to G_{p^5} is $1 + kp^2$. The case in which $\alpha_{21} \not\equiv 0$ in the common G_{p^3} has been covered by the previous argument.

Let next $\alpha_{21} \equiv 0$ in the common G_{p^3} . The groups of order p^3 are then defined by

$$\left. \begin{array}{l} a_{210} = a_{211} = 0 \\ a_{321} = h a_{320} \end{array} \right\}, \quad \left. \begin{array}{l} a_{210} = a_{211} = 0 \\ a_{320} = 0 \end{array} \right\}. \quad (5)$$

But for either of these groups, the operation P is contained in (3_1) , and as before we conclude that the number of conjugates can not be $1 + kp^2$.

(I₃). Let the number of conjugates be $1 + kp^3$. For the common G_{p^2} , we have the following possibilities:

- a) $\alpha_{21} \not\equiv 0$, $\alpha_{32} \not\equiv 0$; b) $\alpha_{21} \equiv 0$, $\alpha_{32} \not\equiv 0$; c) $\alpha_{21} \equiv 0$, $\alpha_{32} \equiv 0$; d) $\alpha_{21} \not\equiv 0$, $\alpha_{32} \equiv 0$.

* Burnside: "Theory of Groups," p. 94, Cor. II.

† *Op. cit.*, p. 97.

The operation P for the G_{p^2} defined in either a) or c) is contained in (3_1) and is therefore permutable with G_{p^5} . In b) we note that G_{p^5} must contain a subgroup G_{p^3} defined by (5_1) or (5_2) . The operation permutable with the G_{p^2} in b) is contained either in (3_1) or (3_2) , and we need only consider the case in which P is in (3_2) . Since under this hypothesis P is not permutable with either of the groups G_{p^3} defined in (5_1) , (5_2) , it will transform either of them into a new G'_{p^3} and $\{G_{p^3}, G'_{p^3}\}$ is of order p^4 with $\alpha_{21} \equiv 0$. This G_{p^4} is not a subgroup of G_{p^5} , but is permutable with it. Hence $\{G_{p^5}, G_4\}$ must be of order p^6 , in contradiction with the hypothesis on the order of H . Hence P must be contained in (3_1) . It will then be permutable with G_{p^5} , contradicting the Frobenius-Burnside Theorem. A similar argument applies to the case d). We now conclude that the number of conjugates to G_{p^5} can not be $1 + kp^3$.

(I₄). By Lemma II the only divisor of the order of G of the form $1 + kp^4$ is $(p^6 - 1)(p^4 - 1)$. If the number of conjugates to G_{p^5} were $1 + kp^4$, the index of H under G would be p , whereas the order of $LF\{3, p^2\}$ is not a divisor of $p!$.

(II). Let H contain a G_{p^5} defined by $l_1 a_{210} + l_2 a_{211} = 0$. We shall prove that such a group H can not contain any operation (β_{ij}) not contained in (3_2) . We first show that if any operation (β_{ij}) has β_{13}, β_{23} not both 0, then H contains an operation (γ_{ij}) having $\gamma_{12} = 0, \gamma_{13} = 0, \gamma_{23} \neq 0$.

a) If $\beta_{12} = \beta_{13} = 0$, while $\beta_{23} \neq 0$, this (β_{ij}) is such an operation.

b) The case $\beta_{12} \neq 0, \beta_{13} = 0$, may be disposed of by replacing (β_{ij}) by its inverse. We need then consider only

c) $\beta_{13} \neq 0$. Transforming G_{p^5} by β_{ij} we get (γ_{ij}) , where

$$\gamma_{11} \equiv \bar{\beta}_{12} \beta_{11} \alpha_{21} + \bar{\beta}_{13} \beta_{11} \alpha_{31} + \bar{\beta}_{13} \beta_{21} \alpha_{32} + 1, \quad (1)$$

$$\gamma_{12} \equiv \bar{\beta}_{12} \beta_{12} \alpha_{21} + \bar{\beta}_{13} \beta_{12} \alpha_{31} + \bar{\beta}_{13} \beta_{22} \alpha_{32}, \quad (2)$$

$$\gamma_{13} \equiv \bar{\beta}_{12} \beta_{13} \alpha_{21} + \bar{\beta}_{13} \beta_{13} \alpha_{31} + \bar{\beta}_{13} \beta_{23} \alpha_{32}, \quad (3)$$

$$\gamma_{21} \equiv \bar{\beta}_{22} \beta_{11} \alpha_{21} + \bar{\beta}_{23} \beta_{11} \alpha_{31} + \bar{\beta}_{23} \beta_{21} \alpha_{32}, \quad (4)$$

$$\gamma_{22} \equiv \bar{\beta}_{22} \beta_{12} \alpha_{21} + \bar{\beta}_{23} \beta_{12} \alpha_{31} + \bar{\beta}_{23} \beta_{22} \alpha_{32} + 1, \quad (5)$$

$$\gamma_{23} \equiv \bar{\beta}_{22} \beta_{13} \alpha_{21} + \bar{\beta}_{23} \beta_{13} \alpha_{31} + \bar{\beta}_{23} \beta_{23} \alpha_{32}, \quad (6)$$

$$\gamma_{31} \equiv \bar{\beta}_{32} \beta_{11} \alpha_{21} + \bar{\beta}_{33} \beta_{11} \alpha_{31} + \bar{\beta}_{33} \beta_{21} \alpha_{32}, \quad (7)$$

$$\gamma_{32} \equiv \bar{\beta}_{32} \beta_{12} \alpha_{21} + \bar{\beta}_{33} \beta_{12} \alpha_{31} + \bar{\beta}_{33} \beta_{22} \alpha_{32}, \quad (8)$$

$$\gamma_{33} \equiv \bar{\beta}_{32} \beta_{13} \alpha_{21} + \bar{\beta}_{33} \beta_{13} \alpha_{31} + \bar{\beta}_{33} \beta_{23} \alpha_{32} + 1, \quad (9)$$

where $\bar{\beta}_{ij}$ is the adjoint minor of β_{ji} in $|\beta_{ji}|$.

We wish to show that we can make $\gamma_{12} = \gamma_{13} = 0$, $\gamma_{23} \neq 0$ by properly choosing α_{21} , α_{31} , α_{32} . We can by a normalization of the operator (β_{ij}) make $\bar{\beta}_{12} \neq 0$ without disturbing β_{13} ; viz., by multiplying (β_{ij}) on the left by $B_{32\lambda}$. Now suppose that $\gamma_{23} = 0$. Then, setting $\alpha_{32} = 0$, we have the following equations:

$$\bar{\beta}_{22} \alpha_{21} + \bar{\beta}_{23} \alpha_{31} = 0, \quad (6')$$

$$\bar{\beta}_{32} \alpha_{21} + \bar{\beta}_{33} \alpha_{31} = 0, \quad (9')$$

$$\bar{\beta}_{12} \alpha_{21} + \beta_{13} \alpha_{31} = 0. \quad (3')$$

Multiplying (6') by β_{32} , (9') by β_{33} and (3') by β_{31} , adding and applying $\Delta = 1$, we get $\alpha_{31} = 0$. This is impossible if $\bar{\beta}_{12} \neq 0$. It follows, then, that the operation (γ_{ij}) fulfils the required conditions if $\bar{\beta}_{13} \neq 0$. If $\bar{\beta}_{13} = 0$ we can set $\alpha_{32} = 0$ as before, and now put $\alpha_{21} = 0$, thus making $\gamma_{12} = \gamma_{13} = 0$. Then, if $\gamma_{23} = 0$, we must have equations (6'), (9'), (3') true for arbitrary values of α_{31} . From the previous argument these equations can only be simultaneously true when $\alpha_{31} = 0$. Therefore the operation (γ_{ij}) always has $\gamma_{12} = \gamma_{13} = 0$, $\gamma_{23} \neq 0$.

Transforming $[\alpha]$ by (γ_{ij}) we get an operation

$$\begin{pmatrix} 1 & 0 & 0 \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{pmatrix}, \quad \begin{aligned} \delta_{21} &\equiv \bar{\gamma}_{22} \alpha_{21} + \bar{\gamma}_{23} \alpha_{31} + \bar{\gamma}_{23} \gamma_{21} \alpha_{32}, \\ \delta_{31} &\equiv \bar{\gamma}_{32} \alpha_{21} + \bar{\gamma}_{33} \alpha_{31} + \bar{\gamma}_{33} \gamma_{21} \alpha_{32}. \end{aligned}$$

Setting $\alpha_{21} = 0$, $\alpha_{31} = -\gamma_{21} \alpha_{32}$, then $\delta_{21} = \delta_{31} = 0$ and $\delta_{23} = \gamma_{23}^2 \alpha_{32} \neq 0$ if $\alpha_{32} \neq 0$.

Therefore, by Lemma III, the group H contains every binary transformation of determinant unity on ξ_2, ξ_3 . Hence it contains

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \mu \\ 0 & 0 & 1 \end{pmatrix} \quad (\mu \text{ arbitrary}),$$

and this, together with $B_{31,\lambda} B_{32,\delta}$, generates a group of order p^6 . But this contradicts the hypothesis on the order of H .

(III). By a reflexion* on the left diagonal we may draw the conclusion, with reference to the group G_{p^6} defined by

$$l_3 \alpha_{320} + l_4 \alpha_{321} = 0,$$

that it can not contain any operation (β_{ij}) not contained in (3_3) . H is then

* Dickson, AMERICAN JOURNAL, Vol. XXVIII, p. 9.

multiply isomorphic with the group of binary substitutions on ξ_1, ξ_2 of order pt (t prime to p).

We have now established the following theorem covering the results of the three cases I, II, III.

THEOREM. *Any subgroup H of order $p^5 N$ of the linear ternary homogeneous group G defined by the relation*

$$l_1 a_{210} + l_2 a_{211} + l_3 a_{320} + l_4 a_{321} = 0$$

(in which neither $l_1 = l_2 = 0$ nor $l_3 = l_4 = 0$) contains a self-conjugate subgroup G_{p^5} , and within G H is conjugate with a group of operations given by (3_1) having $\alpha_{22}^3 = 1$, $\alpha_{11} \alpha_{22} \alpha_{33} = 1$.

If H contains a subgroup G_{p^5} defined by the relation $l_1 a_{210} + l_2 a_{211} = 0$ or $l_3 a_{320} + l_4 a_{321} = 0$, H is conjugate within G with a group of operations of the form (3_1) or (3_2) .

Subgroups of Order a Multiple of p^4 .

8. Let H be a subgroup of order $p^4 N$ normalized to contain a subgroup G_{p^4} of G_{p^6} . First, let G_{p^4} be the special group $\{\alpha_{21} = 0\}$. We shall prove that it is self-conjugate under H .

If the number of conjugates to G_{p^4} were $1 + kp$ (k prime to p), then H would contain an operation P permutable with the G_{p^3} common to two G_{p^4} of the conjugate set to which G_{p^4} belongs, but not permutable with G_{p^4} , according to the Frobenius-Burnside Theorem. The operations permutable with the common G_{p^3} are, however, permutable with G_{p^4} . In the same way it follows that there can not be $1 + kp^2$ or $1 + kp^3$ conjugates to G_{p^4} . Finally, there can be no group H containing a G_{p^4} having $1 + kp^4$ conjugates. Indeed, by Lemma II, such a group H would be of index p^2 under G . But the linear homogeneous group G is isomorphic with the linear fractional $LF(3, p^2)$ and, if such an H existed, $LF(3, p^2)$ would be representable as a substitution group on p^2 letters and also its subgroup the linear fractional $LF(2, p^2)$. The latter can not be represented as a substitution on p^2 letters except when $p = 3$ (references in the Introduction). No such subgroup H can exist when $p = 3$, for the order of G when $p = 3$ is not a divisor of $p^2!$. Hence the subgroup $\{\alpha_{21} = 0\}$ is self-conjugate in H .

Next, by a reflexion on the left diagonal we reach the same conclusion with reference to $\{\alpha_{32} \equiv 0\}$.

Let H now be normalized to contain any one of the remaining groups of order p^4 . If the number of conjugates to G_{p^4} is $1 + kp^i$ (k prime to p , $i = 1, 2, 3$), H must* contain an operation P permutable with $G_{p^{4-i}}$ but not with G_{p^4} .

First, let $i = 1$. Then the greatest common subgroup is of order p^3 , and the operation P is contained in one of the sets $(3_1), (3_2), (3_3)$. We shall not consider (3_3) , for the results obtained in case P lies in (3_2) are immediately applicable to the case when P lies in (3_3) by reflexion on the left diagonal. With this agreement the greatest common G_{p^3} of two groups of order p^4 is of one of two types:

- a) $\alpha_{21} \not\equiv 0, \quad \alpha_{32} \not\equiv 0;$
- b) $\alpha_{21} \equiv 0, \quad \alpha_{32} \not\equiv 0.$

The operations permutable with either of the groups a), b) are contained in the set (3_1) by Lemmas II and V, and P therefore is permutable with G_{p^4} , contradicting the Frobenius-Burnside Theorem. The number of conjugates to G_{p^4} under H can not be $1 + kp$.

Next, let $i = 2$ or 3 . Then G_{p^m} , the greatest common subgroup, will be of order p^2 or p . P can not be contained in (3_1) , since that leads to a contradiction of the Frobenius-Burnside Theorem. We consider P in the set (3_2) . Any G_{p^4} of the set $(p, 3)$ contains the subgroup G_{p^2} defined by $\alpha_{21} \equiv \alpha_{32} \equiv 0$. The operations permutable with this G_{p^2} are contained in (3_1) . Hence P in (3_2) is not permutable with G_{p^2} . It transforms G_{p^2} into G'_{p^2} ; and $\{G_{p^2}, G'_{p^2}\}$ is of order p^4 , G_{p^4} having $\alpha_{21} \equiv 0$. Hence $\{G'_{p^4}, G_{p^4}\}$ is of order a power of p higher than p^4 , and this contradicts the hypothesis on H . Hence the number of conjugates to G_{p^4} can not be $1 + kp^2$ or $1 + kp^3$. We have previously shown that the number can not be $1 + kp^4$. Hence G_{p^4} must be self-conjugate under H , and we may state the following

THEOREM. *Every subgroup H , of order a multiple of p^4 , of G contains a self-conjugate group G_{p^4} . Within G , H is conjugate with a group of operations (3_2) or a group of operations (3_3) .*

* Burnside: "Theory of Groups," p. 94, Cor. II, and the theorem on p. 97.

VITA.

I, Robert Lacey Börger, was born in Amherst Court House, Virginia, August, 1873. I attended the public schools and afterward the Florida Agricultural and Mechanical College, Lake City, Florida, receiving the degree A.B. June, 1893.

I studied in the Johns Hopkins University during the year 1894-1895. From 1896 to 1904 I had charge of the department of mathematics in the Florida Agricultural and Mechanical College.

I studied in the University of Chicago during the summer quarters 1898-1903 and the year 1904-1905, receiving the degree A.M. in 1905.

In the University of Chicago I have had courses under Professors Moore, Bolza, Maschke, Dickson, Slaughter, Laves, and Moulton. I desire to thank them, and especially Professor Dickson, for their kindly interest in the progress of my studies.

